Instantons

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Prelude

We will start with the conception of instantons as a Quantum Mechanical notion, and proceed to promote many of these properties into a field theoretic context.

Notation: We will spend most of our time exploring Euclideanized versions of concepts, and will employ a continuation into Minkowski space via the Wick rotation $x^0 = ix^4$, where x^4 will represent the Euclidean time, usually taken from $[-T/2, T/2]$. The Euclidean metric will simply be *diag*(1,1,1,1), and the Minkowski metric *diag*(1,-1,-1,-1), as usual. We will be considering Euclideanized path integrals:

$$\langle x_f | \, e^{-HT} \, | x_i \rangle = N \int \mathcal{D}x \, e^{-S/\hbar} = \sum_n e^{-E_n T/\hbar} \langle x_f | n \rangle \langle n | x_i \rangle \quad (1)$$

where $\mathcal{D}x$ (often written $[dx]$) is the path integral measure.

Why bother?

We bother because I prefer passing this course to the alternative.

Jest aside, note that we wish to consider quantum mechanical results in the context of QFTs. However, one of our primary tools in QFT, perturbation theory, fails to incorporate many of these effects, such as tunnelling, domain walls, flux tubes and the like. Instantons will address some non-perturbative phenomena, and can be applied to the study of tunnelling behaviour and offer insight into quantum corrections to the classical.

What are instantons, and where can I find them?

Instantons are simply solutions to the classical equations of motion that arise from an action describing our theory. They are localised objects, much like solitons, but last only for an instant unlike them, granting them their name (unless one wishes to abide by Polyakov's choice of "pseudoparticles").

We can find them in simple tunnelling phenomena, and given a specific problem, they can be explicitly constructed. Let us do so.

Well well, what do we have here? I

The title may have tipped the pun-inclined to our first topic of consideration: the even double well potential:

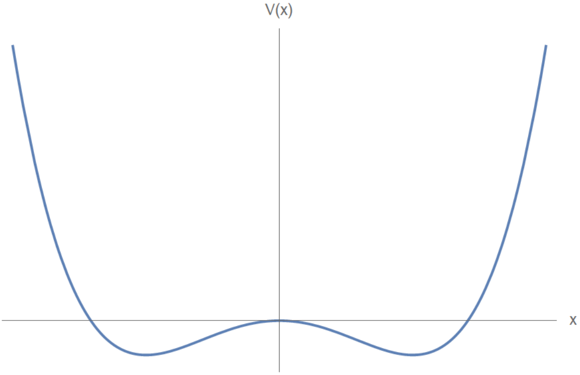

Figure: The Double Well

Well well, what do we have here? II

Let us assume that we find the minima at $x = \pm a$, where the potential is vanishing (it can be shifted by an arbitrary constant if not).
We are interested in tunnelling phenomena, which allow us to move between the minima. Therefore, we must compute the value of the tunnelling amplitude:

$$\langle a| e^{-HT/\hbar} |-a\rangle = \langle -a| e^{-HT/\hbar} |a\rangle \tag{2}$$

This will be done by approximating the functional integral with the semi-classical limit (that contains $\mathcal{O}(\hbar)$ corrections).
Classically- vanishing energy solution to the EoM:

$$\dot{x} = \sqrt{2V} \Leftrightarrow t = t_1 + \int_0^x dx' \frac{1}{\sqrt{2V}} \tag{3}$$

Well well, what do we have here? III

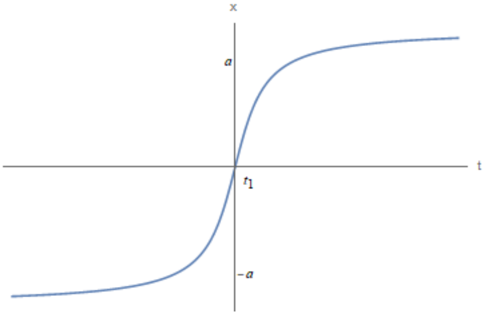

Figure: An Instanton with Centre $t = t_1$

Well well, what do we have here? IV

The action for such an instanton, using (3) is then:

$$S_0 = \int dt \left(\frac{dx}{dt}\right)^2 = \int_{-a}^{a} dx \sqrt{2V} \tag{4}$$

For very large times, $x \sim a$, which implies $\dot{x} \approx \omega(a - x)$, solving approximately to give $(a - x) \propto e^{-\omega t}$.

Instanton size: $\mathcal{O}\left(\frac{1}{\omega}\right)$ - "well" localised in time

We could have tunnelled from a to $-a$ instead: anti-instanton

The Action Principles I

For the classical path $\bar{x}$ between $-a$ and a:
Stationary points of the action, $\frac{\delta S}{\delta \bar{x}} = -\ddot{\bar{x}} + V'(\bar{x}) = 0$.

Path integral: by using $x = \bar{x} + c_n x_n$, for some orthonormal basis of $\{x_n\}$ that are eigenvectors of the extremization operation:

$$-\ddot{x_n} + V''(\bar{x})x_n = \lambda_n x_n$$

The path integral measure is proportional to $\prod_n dc_n$. The orthonormality allows us to make the path integral a product of Gaussians corrected by orders of $\hbar$. Then,

$$\langle -a| \, e^{-HT/\hbar} \, |a\rangle = Ne^{-S[\bar{x}]/\hbar} \prod_n \lambda_n^{-1/2} \left(1 + \mathcal{O}\left(\hbar\right)\right) \qquad (5)$$

The Action Principles II

The product of eigenvalues is the determinant of the operator: $\prod_n \lambda_n^{-1/2} = \det[-\partial_t^2 + V''(\bar{x})]^{-1/2}$

I will, without proof, state the result of such a determinant, with a specific normalisation choice:

$$N^2 \det[-\partial_t^2 + V''(\bar{x})] = \pi\hbar\psi_0(T/2) \tag{6}$$

where ψ_0 is an eigenfunction of the operator with eigenvalue 0 (classical resonant frequency). In the case of the S.H.O (which the instanton approximates roughly over long times), $V''(\bar{x}) = \omega^2$, and:

$$\psi_0 = \frac{1}{\omega} \sinh(\omega t + \omega T/2)$$

The Action Principles III

We can make use of this, since we are considering a large time scale $\omega T \gg 1$. If not for the instanton, we would be stuck at the bottom of a well, so $V'' \approx \omega^2$ may be used. Thus, we may state the result of the determinant approximately, given that the instanton size is $\mathcal{O}\left(1/\omega\right)$:

$$N \det[-\partial_t^2 + \omega^2]^{-1/2} = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \tag{7}$$

which implies

$$\langle a| \, e^{-HT/\hbar} \, |-a\rangle = e^{-S[\bar{x}]/\hbar} \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \left(1 + \mathcal{O}\left(\hbar\right)\right) \tag{8}$$

Comparing this to (1), since the ground state contributes the most for large times, we get $E_0 = \hbar\omega/2$ for the H.O.

Other Soltuions

Is the solution we have identified unique?
No. How about $-a \to a \to -a \to a$?

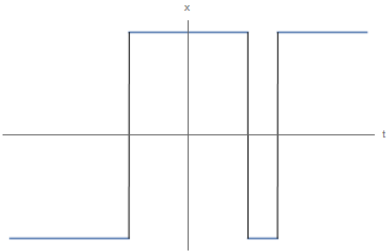

Figure: Other Paths

Since they're well localised in time, the transitions get compressed $\to$ step functions.
These other solutions correspond to approximate stationary points in the action.

The Functional Integration I

Why restrict ourselves to 2 jumps? We may go for n such jumps. Assume wide separation - dilute gas approximation: $S = nS_0$.

By the wide separation, we know that most of the time is spent in a well, not tunnelling $\implies V'' \approx \omega^2$ holds. Thus, the determinant is almost identical, but gets corrected by a factor (call it K) for each instanton. Thus,

$$N \det[-\partial_t^2 + \omega^2]^{-1/2} = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} K^n \tag{9}$$

Path integration: we must be able to integrate over the centres of the instantons:

$$\int_{-T/2}^{T/2} dt_1 \prod_{j=2}^{n} \int_{-T/2}^{t_{j-1}} dt_j = \frac{T^n}{n!}$$

The Functional Integration II

Finally, recall that an instanton tunnels from $-a$ to a, and vice-versa for an anti-instanton. Thus, tunnelling amplitudes consider only n odd. Given this,

$$\langle a| e^{-HT/\hbar} |-a\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \sum_{n\,\mathrm{odd}} \frac{(Ke^{-S_0/\hbar}T)^n}{n!}(1+\mathcal{O}(\hbar)) \tag{10}$$

The sum resolves into a sinh of the arguments raised to the n^{th} power. Had we been looking for the amplitude of staying in a minimum, we would sum over even n, and arrive at a cosh solutions.

Comparing this to (1), we note that the energy eigenvalues are $E_{\pm} = \hbar\omega/2 \pm \hbar Ke^{-S_0/\hbar}$. The instanton correction is then immediately apparent.

The Functional Integration III

Note that K marks the deviation of our potential from being an exact H.O. However, by time translation invariance, the operator $-\partial_t^2 + V''(\bar{x})$ has a zero eigenvalue, corresponding to $x_1 = S_0^{-1/2}\dot{\bar{x}}$.

In this case, $\mathcal{D}x = \prod_n dc_n/(2\pi\hbar)$ will produce a change with respect to changing c_1 as well. The corresponding instanton has a centre at t_1, which we integrated over. Using this, $dx = (d\bar{x}/dt)dt_1$. But if we only change c_1, $dx = x_1 dc_1$. Matching these, in evaluating the determinant, eigenvalues of 0 give a contribution of $(2\pi\hbar)^{-1/2}dc_1/dt_1 = S_0/\sqrt{2\pi\hbar}$. Thus, we get

$$K = \frac{S_0}{\sqrt{2\pi\hbar}}\left|\frac{\det[-\partial_t^2 + \omega^2]}{\det_{\emptyset}[-\partial_t^2 + V''(\bar{x})]}\right|^{1/2} \tag{11}$$

where $\det_{\emptyset}$ skips the 0 eigenvalue. We have computed a QM instanton correction!

A Few Comments I

Loosely speaking, such tunnelling behaviour usually implies instability, which marks an a non-zero imaginary part of E_0, corresponding to the lifetime of the state. In fact, this IS the correction to the energy: $\mathrm{Im}[E_0] = \frac{1}{2}\hbar|K|e^{-S_0/\hbar}$.

The dilute gas approximation modulates this instability. Should instantons not be well separated, we must then account for effects that come from their interactions (i.e. $S \neq nS_0$). This would produce greater contributions to the suppression of the lifetime, but, in return, would be suppressed by the low probability of closely spaced instantons.

Should we be interested in the a potential with infinitely many equally spaced minima, the number of instantons and

A Few Comments II

anti-instantons are not constrained. Thus, to jump to the minimum $|x_i\rangle$ m minima away, from $|x_{i-m}\rangle \equiv |x_k\rangle$,

$$\langle x_i| \, e^{-HT/\hbar} \, |x_m\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \sum_{n,\bar{n}} \frac{(Ke^{-S_0/\hbar}T)^{n+\bar{n}}}{n!\bar{n}!} \delta_{(n-\bar{n})-m} \tag{12}$$

This can be resolved with the identity

$$\delta_{jk} = \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta(j-k)}$$

to produce

$$\langle x_i| \, e^{-HT/\hbar} \, |x_k\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \int_0^{2\pi} e^{im\theta} \frac{d\theta}{2\pi} \exp 2KT \cos\theta e^{-S_0/\hbar} \tag{13}$$

Gauged Field Theories: A Prelude I

Our discussion will primarily focus on $\mathfrak{su}(N)$ algebras for the theories, but the results will can be extended to other Lie algebras corresponding to some compact Lie group G. We use generators corresponding to:

$$[T^a, T^b] = c^{abc} T^c \tag{14}$$

We define the normalised Cartan inner product as:

$$\langle T^a, T^b \rangle = \delta^{ab} \tag{15}$$

For $\mathfrak{su}(2)$, $c^{abc} = \varepsilon^{abc}$, and the isospinor representation presents $T^a = -i\sigma^a/2$. The Cartan inner product is then $-2\mathrm{Tr}[T^a T^b]$.

We further consider the gauge fields corresponding to the generators A^a_μ. It is convenient to deal with them in a fixed T^a

Gauged Field Theories: A Prelude II

basis: $A_\mu \equiv g A_\mu^a T^a$. This also provides us the field strength tensor (FST):

$$F^{\mu\nu} = \partial^{[\mu} A^{\nu]} + [A^\mu, A^\nu]$$

We will consider the free theory, with a Euclidean action of the form:

$$S = \frac{1}{4g^2} \int d^4x \langle F^{\mu\nu}, F^{\mu\nu} \rangle \tag{16}$$

For a complete definition, we must also define gauge transformations, and the affine connection[1] (covariant derivative):

$$D_\sigma F_{\mu\nu} = \partial_\sigma F_{\mu\nu} + [A_\sigma, F_{\mu\nu}]$$

A gauge transformation is a map $\Omega : R^4 \to G$, or more generally, with a space-time manifold M, $\Omega : M \to G$. In our algebra, this is realised by the exponential map, and can be made explicit

Gauged Field Theories: A Prelude III

$$\Omega(x) = \exp\left(\alpha^a(x) T^a\right) \tag{17}$$

Under such a transformation, we have:

$$A_\mu \to \Omega(A_\mu + \partial_\mu)\Omega^{-1}, \quad F_{\mu\nu} \to \Omega F_{\mu\nu} \Omega^{-1} \tag{18}$$

A vanishing $F^{\mu\nu}$ implies that A_μ is a gauge transform of 0, and can be expressed as $\Omega \partial_\mu \Omega^{-1}$. Given a field that transforms as $\phi \to \Omega\phi$, we express the covariant derivative explicitly:

$$D_\mu \phi = \partial_\mu \phi + A_\mu \phi$$

[1] This doesn't require a prescribed metric, but given one, we may construct a frame bundle with a connection.

The Finite Action

We will study field configurations that leave the action finite, to be able to find semi-classical approximations to the path integral ($e^{-S/\hbar}$). Gaussian approximations centred around infinite configurations gives us a vanishing result.

To keep the action finite at large $r \equiv |x|$, $F_{\mu\nu}$ must go as $\mathcal{O}\left(1/r^3\right)$ at most, as a dimensional analysis of (16) would tell us. It must then go to 0 at the boundary. However, the same need not be true for A_μ, since it can simply be a gauge transformation of 0. Thus, for some Ω, $A_\mu = \Omega\partial_\mu\Omega^{-1} + \mathcal{O}\left(1/r^2\right)$.

Thus, each finite action gauge configuration corresponds to an Ω: a map from $S^3 \rightarrow G$, where S^3 is understood as the spherical boundary of the space-time manifold at the radial infinity.

The Homotopy of the Configuration I

The choice of the configuration mapping is not gauge-invariant. We may employ another gauge transformation, Υ, and transform A_μ as per equation (18). This is equivalent to $\Omega \to \Upsilon\Omega + \mathcal{O}\left(1/r^2\right)$.

However, $\Upsilon = \Omega^{-1}$ isn't always possible. This gauge transformation must be a continuous function in orthogonal radial slices over $r = 0$ to $r \to \infty$. At the origin, Υ must be independent of angular variables, i.e. a constant (set it to 1, and any other constant is a trivial transformation away). Thus, any acceptable configuration of infinity must be continuous deformation of $\Upsilon = 1$, i.e. homotopic to the constant map.
Thus, we can only map a finite action gauge configuration to another within the same homotopy class.

An Example: $\mathfrak{su}(2)$

Hereafter, I will not differentiate between the algebra and the group SU(2).

SU(2) can be represented as unitary unit-determinant matrices, and can be parametrized completely by:

$$\Omega = a + i b_k \sigma^k \tag{19}$$

with $a^2 + b^2 = 1$. Thus, SU(2) is homeomorphic to S^3, and our study will pertain to homotopy classes of mappings $S^3 \to S^3$. Let us define some maps to parametrize the entire homotopy class:

1. Trivial map: $\Omega^{(0)}(x) = 1$
2. Identity map: $\Omega^{(1)}(x) = (x_4 + i x_k \sigma^k)/r$
3. "ν-map": $\Omega^{(\nu)}(x) = \Omega^{(1)}(x)^{\nu}$

A Circular Argument I

It is hard to identify or visualise this for a hyper-sphere, so let us consider the U(1) analogue to these maps. Since U(1) is homeomorphic to S^1, the "ν-map" is $\gamma^{(\nu)}(\theta) = e^{i\nu\theta}$, for $\theta \in (-\pi, \pi]$.
Any map from S^1 to S^1 can be continuously deformed to a set number of "windings" around the origin:

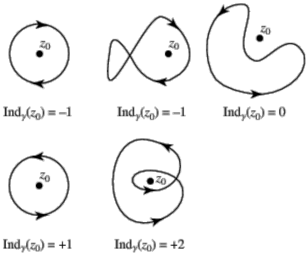

Figure: Winding Number of Maps to S^1

A Circular Argument II

The winding number in the U(1) case can also be thought of as the number of times one wraps around a pole at a point; i.e. for some contour $\mathcal{C}$ expressed by a map $S^1 \to S^1$,

$$\nu = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{dz}{z - 0}$$

We may reparametrize this integral for $\gamma \in \{\gamma^{(\nu)}\}$ to put it in the U(1) context:

$$\nu = \frac{i}{2\pi} \int_0^{2\pi} d\theta \, \gamma \frac{d}{d\theta} \gamma^{-1} \tag{20}$$

If we deform the map γ slightly, by $\delta\gamma$, the $\delta\nu$ corresponds to the integral of $\delta(\gamma \frac{d}{d\theta} \gamma^{-1}) = -i \frac{d\delta\gamma}{d\theta}$, a vanishing quantity. Any number of continuous deformations will leave ν unchanged.

$\mathfrak{su}(2)$ I

The winding number in the higher dimensional case is given by:

$$\nu = \frac{1}{48\pi^2} \int d\theta_1 d\theta_2 d\theta_3 \, \varepsilon^{ijk} \langle \Omega \partial_i \Omega^{-1}, \Omega \partial_j \Omega^{-1} \Omega \partial_k \Omega^{-1} \rangle \tag{21}$$

where the integration is over angles parametrizing[2] S^3. For the trivial map, clearly the integrand vanishes $\partial_i 1 = 0$, and $\nu = 0$ is restored. In the $\nu = 1$ case, $\Omega \partial_i \Omega^{-1} = -i\sigma^i$. Further, $Tr[\varepsilon^{ijk}\sigma^i\sigma^j\sigma^k] = -12$, which implies a Cartan inner product of $(-2) \cdot (-12) = 24$. The area of a unit 3-sphere is $2\pi^2$. Thus, $\Omega^{(1)}$ does correspond to $\nu = 1$.

If $\Omega = \Omega_1 \Omega_2$, $\nu = \nu_1 + \nu_2$. This can be understood in the U(1) case as $e^{im\theta} e^{in\theta} = e^{i(n+m)\theta}$. This gives us the rest of the winding numbers inductively. Thus, we may label each homotopy class by the winding number.

$\mathfrak{su}(2)$ II

Let us now consider a different expression for the winding number.

The Hodge dual is given by $\tilde{F}_{\rho\sigma} \equiv \frac{1}{2}\varepsilon_{\rho\sigma\mu\nu}F^{\mu\nu}$. This is another term that could have been placed in the Lagrangian density. To see why not, define $G^{\mu} \equiv 2\varepsilon^{\mu\nu\rho\sigma}\langle A_{\nu}, \partial_{\rho}A_{\sigma} + \frac{2}{3}A_{\rho}A_{\sigma}\rangle$. It is easy to compute: $\partial_{\mu}G^{\mu} = \langle F, \tilde{F}\rangle$, which is a total divergence and would naively be removed. However, we may rework (21) to give

$$\nu = \frac{1}{32\pi^2}\int d^4x\langle F, \tilde{F}\rangle \tag{22}$$

Clearly, this term (θ term in QCD) contains some important information about the configuration of our gauge. We conclude the treatment of homotopy classes in $\mathfrak{su}(2)$ with a small goody: as long as the Cartan inner product is normalised to the identity, all of the computations apply to any simple Lie group.

[2]Changing parametrizations, the Jacobian is cancelled by the ε determinant.

Gauged QFT Vacua I

We now promote our classical discussion to a quantized formulation.

First, we wish to continue our treatment of finite action configurations. Let us work in a box of size $(T, V \equiv L^3)$, and in the axial gauge $A_3 = 0$. To find the restrictions on allowed configurations, consider the surface term in the variation of the action upon varying A^μ:

$$\delta S = \int d^3 S n^\mu F_{\mu\nu} \delta A^\nu + \dots$$

where $d^3 S n^\mu$ is the oriented surface element of the box. By the antisymmetric nature of F, the normal part of A^μ doesn't contribute.

Gauged QFT Vacua II

The tangential components should match the axial gauge condition and must be kept finite as $V, T \to \infty$.

In other words, the configuration should belong to a fixed homotopy class as we take the box to infinity. This is the only remnant of the boundary conditions of the now infinite box. Thus, we may dispense with the box altogether, and simply integrate over a fixed winding number, "n":

$$F(V, T, n) = N \int \mathcal{D}A^{\mu} e^{-S} \delta_{\nu n} \tag{23}$$

where F is some transition matrix element between states (determined by boundary conditions).

Gauged QFT Vacua III

For large times T_1 and T_2, we expect contributions from different winding numbers:

$$F(V, T_1 + T_2, V, n) = \sum_{n_1+n_2=n} F(V, T_1, n_1)F(V, T_2, n_2) \qquad (24)$$

where n in a large box may come from one configuration class n_1 in one part of the box, and $n - n_1$ in the other, since the winding number relates to the integration of a local (θ term) density.

However, we expect a simple exponential element for a single energy eigenstate, as equation (1) suggests. Over several large time slices, we expect a product of these objects, and not a convolution. There is an easy way to deconvolve: Fourier transforms:

$$F(V, T, \theta) \equiv \sum_n e^{in\theta} F(V, T, n) = N \int \mathcal{D}A^\mu e^{-S} e^{i\nu\theta} \qquad (25)$$

Gauged QFT Vacua IV

We may now identify $F(V, T, \theta)$ as an expectation of e^{-HT} for some energy eigenstate, $|\theta\rangle$, the θ-vacuum:

$$F(V, T, \theta) \propto \langle\theta| \, e^{-HT} \, |\theta\rangle = N' \int \mathcal{D}A^{\mu} e^{-S} e^{i\nu\theta} \tag{26}$$

This implies the importance of an inclusion of the θ term to the Lagrangian density.

Gauged QFT Instantons I

We may now use the knowledge of the vacuum as well as the tunnelling principles we had discussed quantum mechanically to calculate some basic quantities, such as the vacuum expectation value and energy.

Let each instanton action be denoted S_0, a finite quantity. We will consider approximate solutions as we did in the case of the infinite minima potential, widely separated, with n instantons and $\bar{n}$ anti-instantons. These approximate solutions have $\nu = n - \bar{n}$. We may then use (12):

$$\langle\theta| e^{-HT} |\theta\rangle \propto \sum_{n,\bar{n}} \frac{(VTKe^{-S_0})^{n+\bar{n}}}{n!\bar{n}!} e^{i(n-\bar{n})\theta} = \exp 2KVTe^{-S_0} \cos\theta \tag{27}$$

We may then find the energy density of the vacuum by comparing this to (1):

Gauged QFT Instantons II

$$\frac{E(\theta)}{V} = -2Ke^{-S_0} \cos\theta \tag{28}$$

This shows us that the vacua are indeed distinct, and possess different energy densities.

Let us attempt to compute the vev of the θ operator: $\langle\theta| \langle F(x), \tilde{F}(x)\rangle |\theta\rangle$. Since this is a translational invariant, we may simply average over all space. We have observed that the path integral (with explicit normalisation) (cf. (26)) is given by:

$$\frac{1}{VT}\int d^4x \langle\theta| \langle F(x), \tilde{F}(x)\rangle |\theta\rangle = \frac{32\pi^2}{VT}\frac{\int \mathcal{D}A^\mu\, \nu e^{-S} e^{i\nu\theta}}{\int \mathcal{D}A^\mu\, e^{-S} e^{i\nu\theta}} \tag{29}$$

Gauged QFT Instantons III

We may rephrase this as a logarithmic derivative to make the computation easier to process:

$$\langle\theta| \langle F(x), \tilde{F}(x)\rangle |\theta\rangle = -\frac{32\pi^2 i}{VT}\frac{d}{d\theta}\ln\left(\int \mathcal{D}A^\mu\, e^{-S} e^{i\nu\theta}\right) \tag{30}$$

The argument of the logarithm is a known quantity, and is given in (27). Thus, the expectation of the operator is:

$$\langle\theta| \langle F(x), \tilde{F}(x)\rangle |\theta\rangle = -64\pi^2 i K e^{-S_0} \sin\theta \tag{31}$$

With this, we conclude that these distinct vacua all correspond to a different vev of the θ operator, and gathers a contribution from the instantons.

Closing Comments I

We left the instanton action in without computing it. How is this found?
Cauchy-Schwarz inequality: $\langle u, v\rangle \leq \sqrt{\langle u, u\rangle\langle v, v\rangle}$

$$\implies \int d^4x\langle F, F\rangle = \sqrt{\int d^4x\langle F, F\rangle \int d^4x\langle \tilde{F}, \tilde{F}\rangle} \geq \left|\int d^4x\langle F, \tilde{F}\rangle\right|$$

$$S \geq \frac{1}{4g^2}|32\pi^2\nu|$$

The equality is asserted iff $F = \pm\tilde{F}$, a first order ODE. These inequalities can be used to find minimal action configurations in a homotopy class. For example $S_0 = 8\pi^2/g^2$.

When we discussed gauge configurations, we spoke of summing over different winding numbers in different space-time sectors. However, we have neglected contributions on the boundary of each

Closing Comments II

part, where the action might be especially dense. A small argument for this is that taking the box to infinity suppresses these boundary terms relative to the bulk of the manifold.

Instantons have many other uses such as the computation in vacuum energy shifts given charge configurations, explaining the mass of the η' (QCD axial current anomaly), and simply understanding the vacuum structure of theories such as QCD.

Thank you

Thank you for your patience during this presentation. It has been a weird quarter, but an awesome course.

References

1. Weisstein, Eric W. "Contour Winding Number." From MathWorld–A Wolfram Web Resource.
2. S. Coleman, "Aspects of Symmetry," doi:10.1017/CBO9780511565045